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Characterization of exponential distribution via regression of one record value on two non-adjacent record values

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Abstract We characterize the exponential distribution as the only one which satisfies a regression condition. This condition involves the regression function of a fixed record value given two other record values, one of them being previous and the other next to the fixed record value, and none of them are adjacent. In particular, it turns out that the underlying distribution is exponential if and only if given the first and last record values, the expected value of the median in a sample of record values equals the sample midrange.

Keywords characterization · exponential distribution · record values · median · midrange

1 Introduction

In 2006, on a seminar at the University of South Florida, Moe Ahsanullah posed the question about characterizations of probability distributions based on regression of a fixed record value with two non-adjacent (at least two spacings away) record values as covariates. We address this problem here.

To formulate and discuss our results we need to introduce some notation as follows. Let $X_1, X_2, \ldots$ be independent copies of a random variable X with absolutely continuous distribution function $F(x)$. An observation in a discrete time series is called a (upper) record value if it exceeds all previous observations, i.e., X_j is a (upper) record value if $X_j > X_i$ for all $i < j$. If we define the sequence $\{T_n, n \geq 1\}$ of record times by $T_1 = 1$ and $T_n = \min\{j : X_j > X_{T_{n-1}}, j > T_{n-1}\}$, $(n > 1)$, then the corresponding record values are $R_n = X_{T_n}$, $n = 1, 2, \ldots$ (see Nevzorov (2001)).

Let $F(x)$ be the exponential distribution function

$$F(x) = 1 - e^{-c(x - l_F)}, \qquad (x \geq l_F > -\infty), \tag{1}$$

where $c > 0$ is an arbitrary constant. Let us mention that (1) with $l_F > 0$ appears, for example, in reliability studies where l_F represents the guarantee time; that is, failure cannot occur before l_F units of time have elapsed (see Barlow and Proschan (1996), p.13).

We study characterizations of exponential distributions in terms of the regression of one record value with two other record values as covariates, i.e., for $1 \leq k \leq n-1$ and $r \geq 1$ we examine the regression function

$$E[\psi(R_n)|R_{n-k} = u, R_{n+r} = v], \qquad (v > u \geq l_F),$$

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where ψ is a function that satisfies certain regularity conditions. Let $\overline{f}_{u,v}$ denote the average value of an integrable function $f(x)$ over the interval from $x = u$ to $x = v$, i.e.,

$$\overline{f}_{u,v} = \frac{1}{v-u}\int_u^v f(t)dt.$$

Yanev et al. (2008) prove, under some assumptions on the function g, that if F is exponential then for $1 \le k \le n-1$ and $r \ge 1$,

$$E\left[\frac{g^{(k+r-1)}(R_n)}{k+r-1}\Big| R_{n-k} = u, R_{n+r} = v\right] = \binom{k-1+r-1}{k-1}\frac{\partial^{k+r-2}}{\partial u^{r-1}\partial v^{k-1}}\left(\overline{g'}_{u,v}\right), \tag{2}$$

where $v > u \ge l_F$ and g' is the derivative of g. Bairamov et al. (2005) study the particular case of (2) when both covariates are adjacent (one spacing away) to R_n. They prove, under some regularity conditions, that if $k = r = 1$, then (2) is also sufficient for F to be exponential. That is, F is exponential if and only if

$$E\left[g'(R_n)\Big| R_{n-1} = u, R_{n+1} = v\right] = \overline{g'}_{u,v}, \qquad (v > u \ge l_F).$$

Yanev et al. (2008) consider the case when only one of the two covariates is adjacent to R_n and show that, under some regularity assumptions, F is exponential if and only if (2) holds for $2 \le k \le n-1$ and $r = 1$, i.e.,

$$E\left[\frac{g^{(k)}(R_n)}{k}\Big| R_{n-k} = u, R_{n+1} = v\right] = \frac{\partial^{k-1}}{\partial v^{k-1}}\left(\overline{g'}_{u,v}\right), \qquad (v > u \ge l_F).$$

Here we address the case when both covariates are non-adjacent to R_n, which turns to be more complex. Denote for $x \ge l_F$,

$$H(x) = -\ln(1-F(x)) \qquad \text{and} \qquad h(x) = H'(x),$$

i.e., $H(x)$ is the cumulative hazard function of X and $h(x)$ is its hazard (failure) rate function. In this paper, under some additional assumptions on the hazard rate $h(x)$ and the function $g(x)$, we extend the results in Bairamov et al. (2005) to the case when both covariates are non-adjacent. Namely, we shall prove that for fixed $2 \le k \le n-1$ and $r \ge 2$, equation (2) is a necessary and sufficient condition for $F(x)$ to be exponential. Note that the characterization for the non-adjacent case given in Theorem 1B of Yanev et al. (2008) involves, in addition to (2), one more regression condition. We shall show here that (2) alone characterizes the exponential distribution. This result provides a natural generalization of the known special cases mentioned above. As a consequence of our main result, we obtain Corollary 1 below, which seems to be of independent interest with respect to possible statistical applications. Let us also mention that the technique of our proof is different from that used by Dembińska and Wesołowski (2000) in deriving characterization results in terms of regression of a record value on another non-adjacent one.

Further on, for a given continuous function $g(x)$ and positive integers i and j, we denote

$$M(u,v) = \overline{g'}_{u,v} = \frac{g(v)-g(u)}{v-u}, \ {}_iM_j(u,v) = \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(M(u,v)\right), \quad (u \ne v), \tag{3}$$

as well as ${}_iM(u,v)$ and $M_j(u,v)$ for the ith and jth partial derivative of $M(u,v)$ with respect to u and v, respectively.

Theorem *Let n, k, and r be integers, such that $2 \le k \le n-1$ and $r \ge 2$. Assume that $F(x)$ satisfies the following conditions.*

(i) The nth derivative $F^{(n)}(x)$ where $n = \max\{k, r\}$ is continuous in (l_F, ∞);

(ii) $h(x)$ is nowhere constant in a small interval $(l_F, l_F + \varepsilon)$ for $\varepsilon > 0$;

(iii) $h(l_F+) > 0$ and $\left|h^{(n)}(l_F+)\right| < \infty$ for $n \le \max(2, r-1)$.

Suppose the function $g(x)$ satisfies

(iv) $g(x)$ is continuous in (l_F,∞) and $g^{(k+r-1)}(x)$ is continuous in (l_F,∞);
(v) ${}_{r-1}M_k(l_F+,v)\neq 0$ for $v\geq l_F$;
(vi) if $r=2$ then $|g^{(k+2)}(l_F+)|<\infty$, and if $r\geq 3$ then $|g^{(k+2r-1)}(l_F+)|<\infty$.
Then (2) holds if and only if X has the exponential distribution (1) with $c=h(l_F+)$.

Remark. I conjecture that the assumption (vi) can be weakened to $|g^{(k+r)}(l_F+)|<\infty$ for any $r\geq 2$, retaining the symmetry with respect to k and r from the case $r=2$. One can verify this in the case $r=3$ by extending the approximation formula in Lemma 4.

We refer to Leemis (1995) for distributions, related to reliability and lifetime modeling, whose hazard functions satisfy the assumptions (ii) and (iii). Also the two corollaries below provide examples of functions $g(x)$ which satisfy the assumptions of the Theorem.

We continue with two interesting particular choices for $g(x)$. First, setting

$$g(x)=\frac{x^{k+r}}{(k+r)!}\quad\text{and thus}\quad\frac{g^{(k+r-1)}(x)}{k+r-1}=\frac{x}{k+r-1},$$

one can see that the assumptions (iv)-(vi) of the Theorem are satisfied and

$$\binom{k+r-2}{k-1}\,{}_{r-1}M_{k-1}(u,v)=\frac{1}{k+r-1}\frac{ru+kv}{k+r}.$$

Therefore, we obtain the following corollary.

Corollary 1 *Let n, k, and r be integers, such that $2\leq k\leq n-1$ and $r\geq 2$. Suppose assumptions (i)-(iii) of the Theorem hold. Then X has the exponential distribution (1) with $c=h(l_F+)$ if and only if*

$$E[R_n|R_{n-k}=u,R_{n+r}=v]=\frac{ru+kv}{k+r},\qquad(v>u\geq l_F).\tag{4}$$

Note that the right-hand side of (4) is a weighted average of the two covariate values - each covariate being given weight proportional to the number of spacings R_n is away from the other covariate. In particular, (4) with $k=r$ becomes

$$E[R_n|R_{n-k}=u,R_{n+k}=v]=\frac{u+v}{2},\qquad(2\leq k\leq n-1).$$

This last equation allows the following interpretation. Suppose we observe $2n-1$ record values $R_1,\ldots,R_{2n-1}$ where $n\geq 2$. Then X is exponential if and only if, given the first and last record values, the expected value of the median R_n in the sample equals the sample midrange.

We continue with another choice of $g(x)$ from (2). Let $l_F>0$ and

$$g(x)=\frac{(-1)^{k+r-1}}{(k+r-1)!}\,\frac{1}{x}\quad\text{and thus}\quad\frac{g^{(k+r-1)}(x)}{k+r-1}=\frac{1}{(k+r-1)x^{k+r}}.$$

It is not difficult to see that the assumptions (iv)-(vi) of the Theorem are satisfied and

$$\binom{k+r-2}{k-1}\,{}_{r-1}M_{k-1}(u,v)=\frac{1}{(k+r-1)u^rv^k}.$$

Hence, the Theorem implies the following result.

Corollary 2 *Let n, k, and r be integers, such that $2\leq k\leq n-1$ and $r\geq 2$. Suppose assumptions (i)-(iii) of the Theorem hold. Then X has the exponential distribution (1) with $c=h(l_F+)$ if and only if*

$$E\left[\frac{1}{R_n^{k+r}}\middle|R_{n-k}=u,R_{n+r}=v\right]=\frac{1}{u^rv^k},\quad(v>u\geq l_F>0).\tag{5}$$

Finally, let us mention that, following Bairamov et al. (2005), one can obtain an extension of the Theorem that involves monotone transformations of X, see also Yanev et al. (2008), Theorem 3. Consequently, the characterization examples given in the above two papers can be modified for the case of non-adjacent covariates.

2 Preliminaries

In this section we present four technical lemmas, which we use in Section 3 to prove the Theorem. First, we prove an identity that links the derivatives of $g(x)$ with those of $M(u,v) = (g(v) - g(u))/(v-u)$. Denote $(n)_{(m)} = n(n-1)\dots(n-m+1)$ $(m \geq 1)$; $n_{(0)} = 1$.

Lemma 1 For any positive integer k and $n \geq 2$

$$(n-1)!g^{(k+n-1)}(v) = \sum_{i=0}^{n} \binom{n}{i} (k+n-1)_{(n-i)}(v-u)^i \; {}_{n-1}M_{k-1+i}(u,v), \qquad (v > u). \quad (6)$$

Proof. For simplicity write ${}_iM_j$ for ${}_iM_j(u,v)$. According to Lemma 1 in Yanev et al. (2008), we have for $i, j \geq 1$

$$g^{(j)}(v) = (v-u)M_j + jM_{j-1}, \quad i\; {}_{i-1}M_j = (v-u)\; {}_iM_j + j\; {}_iM_{j-1}, \qquad (v > u). \qquad (7)$$

To prove (6) we use induction with respect to n. Referring to (7), we have

$$\begin{aligned} g^{(k+1)}(v) &= (v-u)M_{k+1} + (k+1)M_k \\ &= (v-u)[(v-u)\; {}_1M_{k+1} + (k+1)\; {}_1M_k] + (k+1)[(v-u)\; {}_1M_k + k\; {}_1M_{k-1}] \\ &= (k+1)k\; {}_1M_{k-1} + 2(k+1)(v-u)\; {}_1M_k + (v-u)^2\; {}_1M_{k+1}, \end{aligned}$$

which is (6) with $n = 2$. To complete the proof, assuming (6), we need to show that

$$n!g^{(k+n)}(v) = \sum_{i=0}^{n+1} \binom{n+1}{i} (k+n)_{(n+1-i)}(v-u)^i \; {}_nM_{k-1+i}. \qquad (8)$$

Differentiating both sides of (6) with respect to v and multiplying by n, we obtain

$$n!g^{(k+n)}(v) = \sum_{i=0}^{n} \binom{n}{i} (k+n-1)_{(n-i)} n \left[i(v-u)^{i-1}\; {}_{n-1}M_{k+i-1} + (v-u)^i\; {}_{n-1}M_{k+i} \right]. \quad (9)$$

Applying the second formula in (7) repeatedly, we have

$$\begin{aligned} & n\left[i(v-u)^{i-1}\; {}_{n-1}M_{k+i-1} + (v-u)^i\; {}_{n-1}M_{k+i} \right] \\ &= i(v-u)^{i-1}\left[(v-u)\; {}_nM_{k+i-1} + (k+i-1)\; {}_nM_{k+i-2} \right] \\ &\quad + (v-u)^i \left[(v-u)\; {}_nM_{k+i} + (k+i)\; {}_nM_{k+i-1} \right] \qquad (10) \\ &= (v-u)^{i+1}\; {}_nM_{k+i} + (k+2i)(v-u)^i\; {}_nM_{k+i-1} + i(k+i-1)(v-u)^{i-1}\; {}_nM_{k+i-2}. \end{aligned}$$

Therefore, by (9) and (10), we have

$$\begin{aligned} n!g^{(k+n)}(v) &= \sum_{i=0}^{n} \binom{n}{i} (k+n-1)_{(n-i)}(v-u)^{i+1}\; {}_nM_{k+i} \\ &\quad + \sum_{i=0}^{n} \binom{n}{i} (k+n-1)_{(n-i)}(k+2i)(v-u)^i\; {}_nM_{k+i-1} \\ &\quad + \sum_{i=0}^{n} \binom{n}{i} (k+n-1)_{(n-i)} i(k+i-1)(v-u)^{i-1}\; {}_nM_{k+i-2} \qquad (11) \\ &= S_1 + S_2 + S_3, \qquad \text{say.} \end{aligned}$$

Changing the summation index to $l = i+1$ we obtain

$$S_1 = \sum_{l=0}^{n+1} \binom{n}{l-1} (k+n-1)_{(n-l+1)}(v-u)^l\; {}_nM_{k+l-1} \qquad (12)$$

and setting $l = i-1$, we have

$$S_3 = \sum_{l=0}^{n-1} \binom{n}{l+1} (l+1)(k+n-1)_{(n-l-1)}(k+l)(v-u)^l \ {}_nM_{k+l-1}, \tag{13}$$

assuming $\binom{n}{l} = 0$ for $l = -1$ or $l > n$. Now, observing that

$$\begin{aligned}
&\binom{n}{i-1}(k+n-1)_{(n-i+1)} + \binom{n}{i}(k+n-1)_{(n-i)}(k+2i) + \binom{n}{i+1}(k+n-1)_{(n-i-1)}(i+1)(k+i) \\
&\quad = (k+n-1)_{(n-i)} \left[\binom{n}{i-1}(k+i-1) + \binom{n}{i}(k+2i) + \binom{n}{i+1}(i+1) \right] \\
&\quad = \binom{n+1}{i}(k+n)_{(n-i+1)},
\end{aligned} \tag{14}$$

one can see that (11)-(14) imply (8) which completes the proof of the lemma.

For simplicity, further on we denote, for integer $i, j \geq 0$ and $v \geq l_F$,

$${}_iM_j(v) = {}_iM_j(l_F+, v).$$

The following result holds.

Lemma 2 *If $|g^{(i+j+1)}(l_F+)| < \infty$ for any non-negative integers i and j, then*

$$\lim_{v \to l_F+} \binom{i+j}{i} \ {}_iM_j(v) = \frac{g^{(i+j+1)}(l_F+)}{i+j+1}. \tag{15}$$

Also, if $|g^{(i+j+1-m)}(l_F+)| < \infty$ for $m = 1, 2, \ldots$, then

$$\lim_{v \to l_F+} (v - l_F)^m \ {}_iM_j(v) = 0 \tag{16}$$

Remark. Note that for $i = k-1$ and $j = r-1$, (15) implies that the limit of the right-hand side of (2) as $v \to l_F+$ equals $g^{(k+r-1)}(l_F+)/(k+r-1)$.

Proof. We use induction with respect to the sum $i+j$. Clearly $\lim_{v \to l_F+} M(v) = g'(l_F+)$. Applying L'Hopital's rule, we have

$$\begin{aligned}
\lim_{v \to l_F+} M_1(v) &= \lim_{v \to l_F+} \frac{g'(v) - M(v)}{v - l_F} \\
&= \lim_{v \to l_F+} g''(v) - \lim_{v \to l_F+} M_1(v).
\end{aligned}$$

Hence, $\lim_{v \to l_F+} M_1(v) = g''(l_F+)/2$. Similarly, $\lim_{v \to l_F+} \ {}_1M(v) = g''(l_F+)/2$. This verifies (15) for $i+j = 0$ and $i+j = 1$. Assuming that (15) is true for $0 \leq i+j \leq n$, we will prove it for $i+j = n+1$. By the second equation in (7) and L'Hopital's rule (the numerator below approaches zero by the induction assumption) we have

$$\begin{aligned}
\lim_{v \to l_F+} \ {}_iM_j(v) &= \lim_{v \to l_F+} \frac{i \ {}_{i-1}M_j(v) - j \ {}_iM_{j-1}(v)}{v - l_F} \\
&= \lim_{v \to l_F+} i \ {}_{i-1}M_{j+1}(v) - j \lim_{v \to l_F+} \ {}_iM_j(v).
\end{aligned}$$

That is,

$$\lim_{v \to l_F+} \ {}_iM_j(v) = \frac{i}{j+1} \lim_{v \to l_F+} \ {}_{i-1}M_{j+1}(v).$$

Iterating, we obtain

$$\lim_{v \to l_F+} \ {}_iM_j(v) = \frac{i!j!}{(i+j)!} \lim_{v \to l_F+} \ M_{j+i}(v). \tag{17}$$

Now, by the first equation in (7) and L'Hopital's rule (the numerator below approaches zero by the induction assumption) we have

$$\begin{aligned}\lim_{v\to l_F+} M_{i+j}(v) &= \lim_{v\to l_F+} \frac{g^{(i+j)}(v)-(i+j)M_{i+j-1}(v)}{v-l_F} \\ &= \lim_{v\to l_F+} g^{(i+j+1)}(v)-(i+j)\lim_{v\to l_F+} M_{i+j}(v)\end{aligned}$$

and hence

$$\lim_{v\to l_F+} M_{i+j}(v)=\frac{1}{i+j+1}g^{(i+j+1)}(l_F+).$$

Substituting this into (17) we complete the proof of the induction step.

Let us now prove (16). Using induction and the second equation in (7), it is not difficult to see that for $m=0,1,\ldots$

$$(v-l_F)^m\ {}_iM_j(v)=\sum_{k=0}^{m}\binom{m}{k}(-1)^k i_{(m-k)}j_{(k)}\ {}_{i-m+k}M_{j-k}(v).$$

Passing to the limit as $v\to l_F+$ and applying (15) we find

$$\begin{aligned}\lim_{v\to l_F+}(v-l_F)^m\ {}_iM_j(v) &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k i_{(m-k)}j_{(k)}\lim_{v\to l_F+}\ {}_{i-m+k}M_{j-k}(v) \\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k i_{(m-k)}j_{(k)}\frac{(i-m+k)!(j-k)!}{(i+j+1-m)!}g^{(i+j+1-m)}(l_F+) \\ &= \frac{i!j!}{(i+j+1-m)!}g^{(i+j+1-m)}(l_F+)\sum_{k=0}^{m}\binom{m}{k}(-1)^k \\ &= 0.\end{aligned}$$

The proof of the lemma is complete.

The next lemma establishes some identities and limit results involving

$$w(v)=\frac{h(v)}{H(v)}(v-l_F),\qquad (v>l_F). \tag{18}$$

Lemma 3 *For $v>l_F$,*

$$\frac{h'(v)}{h(v)}(v-l_F)=\frac{w'(v)}{w(v)}(v-l_F)+w(v)-1. \tag{19}$$

If $F''(v)$ is continuous in (l_F,∞), $h(l_F+)>0$, and $h'(l_F+)\neq 0$, then

$$\lim_{v\to l_F+} w(v)=1,\qquad \lim_{v\to l_F+}\frac{(v-l_F)w'(v)}{w(v)-1}=1, \tag{20}$$

and

$$\lim_{v\to l_F+}\frac{(v-l_F)^2}{w(v)-1}=0. \tag{21}$$

Proof. Differentiating (18) with respect to v, it is not difficult to obtain (19). Applying L'Hopital's rule, we obtain that as $v\to l_F+$

$$\begin{aligned}w'(v) &= \frac{[H(v)h'(v)-h^2(v)](v-l_F)+H(v)h(v)}{H(v)^2} \\ &\sim \frac{[H(v)h''(v)-h(v)h'(v)](v-l_F)+2H(v)h'(v)}{2H(v)h(v)} \\ &\to \frac{h'(l_F+)}{2h(l_F+)}\neq 0.\end{aligned} \tag{22}$$

Now, the continuity of $w(v)$ implies that $w(l_F+) = 1$. It follows by the mean-value theorem and (22) that

$$\lim_{v \to l_F+} \frac{(v - l_F)w'(v)}{w(v) - 1} = \lim_{v \to l_F+} \frac{w'(v)}{w'(\eta)} = 1, \qquad (l_F < \eta < v),$$

i.e., the second limiting result in (20). Finally, applying L'Hopital's rule, it is not difficult to obtain (21). The proof of the lemma is complete.

For positive integers n, r, and k, define the sequence $\{d_n(v)\}_{n=1}^{\infty}$ for $v > l_F$ by the recurrence

$$d_1(v) = \frac{d}{dv}\{\ _{r-1}M_{k-1}(v)H^{n+k-1}(v)\} \quad \text{and} \quad d_{n+1}(v) = \frac{d}{dv}\left\{\frac{d_n(v)}{h(v)}\right\}. \tag{23}$$

In the lemma below, we derive an expansion of $d_n(v)$ in terms of $\ _{r-1}M_j(v)$ and $H^j(v)$ for $k - 1 \le j \le n + k - 1$. Note that if $k > i$, then $\binom{i}{k} = 0$ and $\sum_{j=k}^{i}(\cdot) = 0$.

Lemma 4 *The following identity is true for* $n = 1, 2, \dots$

$$\begin{aligned} d_n(v) = &\sum_{j=0}^{2} \binom{n}{j} (k + n - 1)_{(n-j)}\ _{r-1}M_{k-1+j}(v) \frac{H^{k-1+j}(v)}{h^{j-1}(v)} \\ &- \binom{n}{2} (k + n - 1)_{(n-2)}\ _{r-1}M_k(v) \frac{h'^{k+1}(v)}{h^2(v)} + \sum_{j=3}^{n} c_j(v) H^{k-1+j}(v), \end{aligned} \tag{24}$$

provided that the left and right-hand sides are well-defined.

If $h(l_F+) \neq 0$, $|h^{(n-1)}(l_F+)| < \infty$ *and* $|g^{(k+r+n-1)}(l_F+)| < \infty$ *for* $n = 3, 4, \dots$, *then*

$$\limsup_{v \to l_F+} \left| \sum_{j=3}^{n} c_j(v) H^{k-1+j}(v) \right| < \infty. \tag{25}$$

Proof. Using induction, one can prove that for $n = 1, 2, \dots$

$$\begin{aligned} d_n(v) &= -\frac{h'(v)}{h^2(v)} d_{n-1}(v) + \frac{1}{h(v)} d'_{n-1}(v) \\ &= \sum_{j=1}^{n} c_{j,n}(v) \frac{d^{n-j}}{dv^{n-j}} d_1(v) \end{aligned} \tag{26}$$

and $c_{j,n}(v)$ satisfy the following equations for $j = 2, 3, \dots, n$,

$$c_{j,n}(v) = \frac{1}{h(v)} c'_{j-1,n-1}(v) - \frac{h'(v)}{h^2(v)} c_{j-1,n-1}(v) + \frac{1}{h(v)} c_{j,n-1}(v),$$

where $c_{j,i}(v) = 0$ if $j > i$ and $c_{1,n}(v) = 1/h^{n-1}(v)$. It is not difficult to obtain

$$c_{1,n}(v) = \frac{1}{h^{n-1}(v)}, \qquad c_{2,n}(v) = -\binom{n}{2} \frac{h'(v)}{h^n(v)},$$

and

$$c_{3,n}(v) = \binom{n}{3} \left[\frac{3(n+1)}{4} \frac{(h'(v))^2}{h^{n+1}(v)} - \frac{h''(v)}{h^n(v)} \right].$$

Note that $|c_{j,n}(v)| < \infty$ if $h(v) \neq 0$ and $|h^{(j-1)}(v)| < \infty$ for $1 \le j \le n$.

For simplicity, further on in the proof we drop the left subscript $r-1$ in ${}_{r-1}M_j(v)$ and write $M_j(v)$ instead. Using Leibniz rule for differentiation of the product of two functions, we have for $m \geq 1$

$$\begin{aligned}\frac{d^{m-1}}{dv^{m-1}} d_1(v) &= \frac{d^m}{dv^m}\left\{M_{k-1}(v)H^{n+k-1}(v)\right\}\\ &= \sum_{j=0}^{m}\binom{m}{j} M_{k-1+j}(v)\frac{d^{m-j}}{dv^{m-j}}H^{n+k-1}(v)\end{aligned}$$

and hence (26) becomes

$$\begin{aligned}&d_n(v) = c_{1,n}(v)\\ &\times\left[M_{k-1}(v)\frac{d^n}{dv^n}H^{n+k-1}(v) + nM_k(v)\frac{d^{n-1}}{dv^{n-1}}H^{n+k-1}(v) + \binom{n}{2}M_{k+1}(v)\frac{d^{n-2}}{dv^{n-2}}H^{n+k-1}(v)\right]\\ &\quad + c_{2,n}(v)\left[M_{k-1}(v)\frac{d^{n-1}}{dv^{n-1}}H^{n+k-1}(v) + (n-1)M_k(v)\frac{d^{n-2}}{dv^{n-2}}H^{n+k-1}(v)\right]\\ &\quad + c_{3,n}(v)M_{k-1}(v)\frac{d^{n-2}}{dv^{n-2}}H^{n+k-1}(v) + S(v,M,H), \qquad \text{say.}\end{aligned} \tag{27}$$

The last term, $S(v,M,H)$, in (26) does not include derivatives of $H^{n+k-1}(v)$ of order higher than $n-3$ and it is given by

$$\begin{aligned}S(v,M,H) = &\sum_{j=0}^{2}\left[\sum_{i=j+1}^{n-2+j}\binom{n-2+j}{i}M_{k-1+i}(v)\frac{d^{n-2+j-i}}{dv^{n-2+j-i}}H^{n+k-1}(v)\right]c_{3-j,n}(v)\\ &+\sum_{j=3}^{n}\left[\sum_{i=0}^{n-j}\binom{n-j}{i}M_{k-1+i}(v)\frac{d^{n-j-i}}{dv^{n-j-i}}H^{n+k-1}(v)\right]c_{j+1,n}(v).\end{aligned}$$

Note that $|S(v,M,H)| < \infty$ if $|M_{k-1+n}(v)| < \infty$ and $|c_{j,n}(v)| < \infty$ for $0 \leq j \leq n$.

Recall the formula for the nth derivative of $f^m(v)$ for positive integer m (e.g., Wolfram Research (2009)).

$$\frac{d^n}{dv^n}f^m(v) = \sum_{i_1=0}^{n}\sum_{i_2=0}^{n-i_1}\cdots\sum_{i_{m-1}=0}^{n-\sum_{j=1}^{m-2}i_j}\left(\prod_{p=1}^{m-1}\binom{n-\sum_{j=1}^{p-1}i_j}{i_p}\right)\left(\prod_{j=1}^{m}\frac{d^{i_j}}{dv^{i_j}}f(v)\right),$$

where $i_1, i_2, \ldots, i_m$ is a partition of n. Observe that a term in the right-hand side includes $f^j(v)$ if exactly j of $i_1, \ldots, i_m$ are zeros. Let us apply this formula to $f^m(v) = H^{n+k-1}(v)$. Setting $m = n+k-1$, we see that there are at least $k-1$ zeros in the partition $i_1, i_2, \ldots, i_{n+k-1}$. Also the positions of j zeros among the terms of the partition $i_1, i_2, \ldots, i_{n+k-1}$ can be selected in $\binom{n+k-1}{j}$ ways. Therefore, we can list the terms in the right-hand side, starting with the one that contains $H^{k-1}(v)$, as follows.

$$\begin{aligned}\frac{d^n}{dv^n}H^{n+k-1}(v) = &\binom{n+k-1}{k-1}\binom{n}{1}\cdots\binom{1}{1}(H'(v))^n H^{k-1}(v)\\ &+\binom{n-1}{1}\binom{n+k-1}{k}\binom{n}{2}\binom{n-2}{1}\binom{n-3}{1}\cdots\binom{1}{1}H''(v)(H'(v))^{n-2}H^k(v)\\ &+\binom{n-2}{2}\binom{n+k-1}{k+1}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{1}\binom{n-5}{1}\cdots\binom{1}{1}(H''(v))^2(H'(v))^{n-4}H^{k+1}(v)\\ &+\binom{n-2}{1}\binom{n+k-1}{k+1}\binom{n}{3}\binom{n-3}{1}\binom{n-4}{1}\cdots\binom{1}{1}H'''(v)(H'(v))^{n-3}H^{k+1}(v)\end{aligned}$$

$$+\sum_{j=3}^{n-1} c_j(v,n)H^{k-1+j}(v)$$

$$= (k+1)ka_k h^n(v)H^{k-1}(v) + \binom{n}{2}(k+1)a_k h^{n-2}(v)h'^k(v)$$

$$+\binom{n}{3}a_k\left[\frac{3(n+1)}{4}h^{n-4}(v)(h'^2 + h^{n-3}(v)h''(v)\right]H^{k+1}(v) + \sum_{j=3}^{n-1} c_j(v,n)H^{k-1+j}(v),$$

where $a_k = (n+k-1)!/(k+1)!$ and $c_j(v,n)$ are functions of $h(v)$ and its derivatives. Note that $|c_j(v,n)| < \infty$ if $|h^{(j)}(v)| < \infty$ for $j = 3, \ldots n-1$. Similarly, for the derivatives of $H^{n+k-1}(v)$ of order $n-1$ and $n-2$ we find

$$\frac{d^{n-1}}{dv^{n-1}}H^{n+k-1}(v) = (k+1)a_k h^{n-1}(v)H^k(v) + \binom{n-1}{2}a_k h^{n-3}(v)h'^{k+1}(v)$$
$$+\sum_{j=3}^{n-1} c_j(v,n-1)H^{k-1+j}(v),$$

where $|c_j(v,n-1)| < \infty$ if $|h^{(j-1)}(v)| < \infty$ for $j = 3, \ldots n-1$; and

$$\frac{d^{n-2}}{dv^{n-2}}H^{n+k-1}(v) = a_k h^{n-2}(v)H^{k+1}(v) + \sum_{j=3}^{n-1} c_j(v,n-2)H^{k-1+j}(v),$$

where $|c_j(v,n-2)| < \infty$ if $|h^{(j-2)}(v)| < \infty$ for $j = 3, \ldots n-1$. Using the above three formulas we write (27) as

$$d_n(v)$$
$$= \frac{a_k M_{k-1}(v)}{h^{n-1}(v)}\left[h^n(v)H^{k-1}(v) + \binom{n}{2}(k+1)h^{n-2}(v)h'^k(v)\right.$$
$$\left. + 3\binom{n}{4}h^{n-4}(v)(h'^2 H^{k+1}(v) + \binom{n}{3}h^{n-3}(v)h''^{k+1}(v) + \sum_{j=3}^{n-1} c_j(v,n)H^{k-1+j}(v)\right]$$
$$+\frac{na_k M_k(v)}{h^{n-1}(v)}\left[(k+1)h^{n-1}(v)H^k(v) + \binom{n-1}{2}h^{n-3}(v)h'^{k+1}(v)\right.$$
$$\left.+\sum_{j=3}^{n-1} c_j(v,n-1)H^{k-1+j}(v)\right] + \frac{a_k M_{k+1}}{h^{n-1}(v)}\binom{n}{2}h^{n-2}(v)H^{k+1}(v) + \sum_{j=3}^{n-1} c_j(v,n-2)H^{k-1+j}(v)$$
$$-\binom{n}{2}\frac{a_k M_{k-1}(v)h'(v)}{h^n(v)}\left[(k+1)h^{n-1}(v)H^k(v) + \binom{n-1}{2}h^{n-3}(v)h'^{k+1}(v)\right.$$
$$\left.+\sum_{j=3}^{n-1} c_j(v,n-1)H^{k-1+j}(v)\right] - \binom{n}{2}\frac{a_k(n-1)M_k(v)h'(v)}{h^n(v)}h^{n-2}(v)H^{k+1}(v)$$
$$+\sum_{j=3}^{n-1} c_j(v,n-2)H^{k-1+j}(v) + \binom{n}{3}\left[\frac{3(n+1)}{4}\frac{(h'^2}{h^{n+1}(v)} - \frac{h''(v)}{h^n(v)}\right]a_k M_{k-1}(v)h^{n-2}(v)H^{k+1}(v)$$
$$+\sum_{j=3}^{n-1} c_j(v,n-2)H^{k-1+j}(v) + S(v,M,H)$$
$$= k(k+1)a_k M_{k-1}(v)h(v)H^{k-1}(v) + n(k+1)a_k M_k(v)H^k(v) + \binom{n}{2}\frac{a_k}{h(v)}M_{k+1}(v)H^{k+1}(v)$$
$$-\binom{n}{2}\frac{a_k h'(v)}{h^2(v)}M_k(v)H^{k+1}(v) + \sum_{j=3}^{n-1} b_j(v)H^{k-1+j}(v) + S(v,M,H),$$

where $|b_j(v)| < \infty$ if $h(v) \neq 0$, $|h^{(j)}(v)| < \infty$, and $|M_{k+1}(v)| < \infty$. This is equivalent to (24). The statement in (25) follows from the conditions for finiteness of $\sum_{j=3}^{n-1} b_j(v) H^{k-1+j}(v)$ and $S(v, M, H)$ given in the proof above.

3 Proof of the Theorem

It follows from Lemma 2 in Yanev et al. (2008) that (2) is a necessary condition for X to be exponential. Here we shall prove the sufficiency. The scheme of the proof is as follows: (i) differentiate (2) r times with respect to v, to obtain a differential equation for $H(v)$; (ii) make an appropriate change of variables; (iii) assuming that there is a non-exponential solution, reach a contradiction.

Recall the formula for the conditional density $f_{k,r}(t|u, v)$, say, of R_n given $R_{n-k} = u$ and $R_{n+r} = v$, where $1 \leq k < n$ and $r \geq 1$. Namely, it can be derived using the Markov property of record values (e.g., Ahsanullah (2004), p.6) that for $u < t < v$

$$f_{k,r}(t|u, v) = \frac{(k + r - 1)!}{(k - 1)!(r - 1)!} \frac{(H(t) - H(u))^{k-1}(H(v) - H(t))^{r-1}}{(H(v) - H(u))^{k+r-1}} H'(t). \tag{28}$$

Using (28) we can write (2) as

$$\int_u^v g^{(k+r-1)}(t)(H(t)-H(u))^{k-1}(H(v)-H(t))^{r-1} dH(t) = {}_{r-1}M_{k-1}(u, v)(H(v)-H(u))^{k+r-1}.$$

The continuity of $F(x)$ implies $H(l_F+) = 0$ and hence, letting $u \to l_F+$, we have

$$\int_{l_F}^v g^{(k+r-1)}(t) H^{k-1}(t)(H(v) - H(t))^{r-1} dH(t) = {}_{r-1}M_{k-1}(v) H^{k+r-1}(v).$$

Differentiating the above equation r times with respect to v, dividing by $h(v) > 0$ prior to every differentiation (after the first one), and applying Lemma 4 with $n = r \geq 2$, we obtain

$$\begin{aligned} (r-1)! g^{(k+r-1)}(v) h(v) H^{k-1}(v) &= d_r(v) \\ &= \sum_{j=0}^{2} \binom{r}{j} (k + r - 1)_{(r-j)} \; {}_{r-1}M_{k-1+j}(v) \frac{H^{k-1+j}(v)}{h^{j-1}(v)} \\ &\quad - \binom{r}{2} (k + r - 1)_{(r-2)} \; {}_{r-1}M_k(v) \frac{h'^{k+1}(v)}{h^2(v)} + \sum_{j=3}^{r} c_j(v) H^{k-1+j}(v) \end{aligned} \tag{29}$$

where $c_j(v)$ are as in the statement of Lemma 4. For simplicity, further on in the proof we drop the left subscript $r - 1$ in ${}_{r-1}M_j(v)$ and write $M_j(v)$ instead. Multiplying both sides of (29) by $h^{r-1}(v)(v - l_F)^{r-1}/H^{k+r-1}(v) > 0$ and making the change of variables

$$w(v) = \frac{h(v)}{H(v)}(v - l_F), \qquad (v > l_F),$$

we find (for simplicity we write w for $w(v)$)

$$\begin{aligned} (r - 1)! g^{(k+r-1)}(v) w^{r-1} \frac{h(v)}{H(v)} &= w^{r-1} \frac{h(v)}{H(v)} \sum_{j=0}^{2} \binom{r}{j} (k + r - 1)_{(r-j)} M_{k-1+j}(v) \frac{H^j(v)}{h^j(v)} \\ &\quad - \binom{r}{2} (k + r - 1)_{(r-2)} M_k(v) w^{r-2} \frac{h'(v)}{h(v)} (v - l_F) + S_1(v), \end{aligned} \tag{30}$$

where

$$S_1(v) = w^{r-1} \sum_{j=3}^{r} c_j(v) H^{j-1}(v).$$

Referring to Lemma 1 with $n = r$ and $u = l_F$, we write (30) as

$$\begin{aligned} w^{r-1}\frac{h(v)}{H(v)}\sum_{j=0}^{2}\binom{r}{j}(k+r-1)_{(r-j)}(v-l_F)^j M_{k-1+j}(v) + S_2(v) \\ = w^{r-1}\frac{h(v)}{H(v)}\sum_{j=0}^{2}\binom{r}{j}(k+r-1)_{(r-j)}M_{k-1+j}(v)\frac{H^j(v)}{h^j(v)} \\ -\binom{r}{2}(k+r-1)_{(r-2)}M_k(v)w^{r-2}\frac{h'(v)}{h(v)}(v-l_F) + S_1(v), \end{aligned} \tag{31}$$

where

$$S_2(v) = w^r \sum_{j=3}^{r}\binom{r}{j}(k+r-1)_{(r-j)}(v-l_F)^{j-1}M_{k-1+j}(v).$$

It follows, from (31), after simplifying and rearranging terms, that

$$\begin{aligned} w(w-1)(k+1)rM_k(v) + (w^2-1)\binom{r}{2}M_{k+1}(v)(v-l_F) \\ = -\binom{r}{2}\frac{h'(v)}{h(v)}(v-l_F)M_k(v) + \frac{S_1(v)-S_2(v)}{w^{r-2}(k+r-1)_{(r-2)}}. \end{aligned}$$

Finally, applying (19), we obtain

$$\begin{aligned} w(w-1)(k+1)rM_k(v) + (w^2-1)\binom{r}{2}M_{k+1}(v)(v-l_F) \\ = -\binom{r}{2}\left[\frac{w'(v-l_F)}{w} + w - 1\right]M_k(v) + \frac{S_1(v)-S_2(v)}{w^{r-2}(k+r-1)_{(r-2)}}. \end{aligned} \tag{32}$$

If F is exponential, then $w(v) \equiv 1$. Since the exponential F given by (1) satisfies (2), we have that $w(v) \equiv 1$ is a solution of the above equation. To complete the proof we must show that $w(v) \equiv 1$ is the only solution of (32). Suppose $w(v)$ is a solution of (32) and there exists a value v_1 such that $w(v_1) \neq 1$ and $v_1 > l_F$. We want to reach a contradiction. Since F is twice differentiable, we have that $w(v)$ is continuous with respect to v and hence $w(v) \neq 1$ for v in an open interval around v_1. (For a similar argument see Lemma 3 in Su et al. (2008).) Let

$$v_0 = \inf\{v | w(v) \neq 1\}.$$

Since, by (20), $w(l_F+) = 1$, we have $v_0 \geq l_F$. We shall prove that $v_0 = l_F$. Assume on contrary that $v_0 > l_F$. Then $w(v) = 1$ if $l_F < v \leq v_0$ and integration of (18) implies that $h(v)$ is constant-valued in this interval. This contradicts the assumption (ii). Therefore $v_0 = l_F$ and hence equation (32) holds for all $v > l_F$. Dividing (32) by $w - 1 \neq 0$, we obtain

$$\begin{aligned} w(k+1)rM_k(v) + (w+1)\binom{r}{2}M_{k+1}(v)(v-l_F) \\ = -\binom{r}{2}\left[\frac{w'(v-l_F)}{w(w-1)} + 1\right]M_k(v) + \frac{S_1(v)-S_2(v)}{w^{r-2}(w-1)(k+r-1)_{(r-2)}}. \end{aligned} \tag{33}$$

Passing to the limit as $v \to l_F+$ in the left-hand side of (33), we find

$$\begin{aligned} &\lim_{v\to l_F+}\left[wr(k+1)\ {}_{r-1}M_k(v) + (w+1)\binom{r}{2}\ {}_{r-1}M_{k+1}(v)(v-l_F)\right] \\ &= r(k+1)\lim_{v\to l_F+} w\ {}_{r-1}M_k(v) + \binom{r}{2}\lim_{v\to l_F+}(w+1)(v-l_F)\ {}_{r-1}M_{k+1}(v) \\ &= r(k+1)\ {}_{r-1}M_k(l_F+), \end{aligned} \tag{34}$$

where by (16), $\lim_{v\to l_F+}(v-l_F)\ {}_{r-1}M_{k+1}(v)=0$ provided that $|g^{(k+r)}(l_F+)|<\infty$.

Now we turn to the right-hand side of (33). First, consider the case $r=2$. Since $S_1(v)=S_2(v)=0$, we have for the right-hand side of (33)

$$\lim_{v\to l_F+} -\left[\frac{w'(v-l_F)}{w(w-1)}+1\right]\ {}_1M_k(v) = -2\ {}_1M_k(l_F+) \tag{35}$$

where by the second equation in (20), $\lim_{v\to l_F+}(v-l_F)w'/(w-1)=1$. The equations (34) and (35) imply $2(k+1)\ {}_1M_k(l_F+)=-2\ {}_1M_k(l_F+)$, which is not possible. This proves that $w(v)\equiv 1$ is the only solution of (32) when $r=2$.

Let $r\geq 3$. Consider

$$\lim_{v\to l_F+}\frac{S_2(v)}{w-1}=\lim_{v\to l_F+} w^r\frac{(v-l_F)^2}{w-1}\sum_{j=3}^{r}\binom{r}{j}(k+r-1)_{(r-j)}(v-l_F)^{j-3}\ {}_{r-1}M_{k-1+j}(v). \tag{36}$$

By (21) we have $\lim_{v\to l_F+}(v-l_F)^2/(w-1)=0$. In addition, by Lemma 2 we have that if $|g^{(k+r+2)}(l_F+)|<\infty$, then

$$\limsup_{v\to l_F+}|\ {}_{r-1}M_{k+2}(v)|<\infty \quad\text{and}\quad \lim_{v\to l_F+}(v-l_F)^{j-3}\ {}_{r-1}M_{k-1+j}(v)=0,\quad j=4,5,\dots r.$$

Therefore, under the assumptions of the theorem, the limit in (36) is zero.

Let us now prove that

$$\lim_{v\to l_F+}\frac{S_1(v)}{w-1}=\lim_{v\to l_F+}\frac{H^2(v)}{w-1}\sum_{j=3}^{r}c_j(v)H^{j-3}(v)=0. \tag{37}$$

It is not difficult to see that $\lim_{v\to l_F+}H^2(v)/(w-1)=0$. Indeed, Assumption (ii), (18), and the first part of (20) together imply that $H(v)\sim const.(v-l_F)$, where $const.$ is not zero. The limit assertion now follows from (21). Hence, to prove (37), it is sufficient to establish that the sum in its right-hand side is finite. According to (25) with $n=r$, this is true if $h(l_F+)\neq 0$, $|h^{(r-1)}(l_F+)|<\infty$ and $|g^{(k+2r-1)}(l_F+)|<\infty$, which hold by the assumptions of the theorem.

Taking into account (36) and (37), passing to the limit in (33) as $v\to l_F$, we obtain, similarly to the case $r=2$, that $r(k+1)\ {}_{r-1}M_k(l_F+)=-r(r-1)\ {}_{r-1}M_k(l_F+)$ as $v\to l_F+$. This contradiction proves that $w(v)\equiv 1$ is the only solution of (32). The proof of the theorem is complete.

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